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Nonthermal Radiation from Nonstationary Kerr Black Hole

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A crossing of the positive and negative Dirac energy levels occurs near a nonstationary Kerr black hole. The maximum value of the energy of a particle in the nonthermal radiation depends not only on the dragging velocity, but also the evaporation rate and the event horizon shape of the black hole.

The line element of the nonstationary Kerr space-time represented by the advanced Eddington coordinate is given by (Carmeli and Kaye, 1977):

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right) dv^2 + 2\,dv\,dr - 2\,\frac{2Mra\sin^2\theta}{\rho^2}\,dv\,d\phi$$
$$- 2a\sin^2\theta\,dr\,d\phi + \rho^2\,d\theta^2$$
$$+ \left[(r^2 + a^2) + \frac{2Mra^2\sin^2\theta}{\rho^2}\right]\sin^2\theta\,d\phi^2 \tag{1}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, $M = M(v)$, $a = a(v)$.

The nonzero contravariant components of the metric tensor are

$$g^{00} = \frac{a^2\sin^2\theta}{\rho^2}, \qquad g^{01} = g^{10} = \frac{r^2 + a^2}{\rho^2}, \qquad g^{03} = g^{30} = \frac{a}{\rho^2}, \qquad g^{11} = \frac{\Delta}{\rho^2}$$
$$g^{13} = g^{31} = \frac{a}{\rho^2}, \qquad g^{22} = \frac{1}{\rho^2}, \qquad g^{33} = \frac{1}{\rho^2\sin^2\theta} \tag{2}$$

where $\Delta = r^2 + a^2 - 2Mr$.

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Substituting (2) into the Hamilton–Jacobi equation (Damour, 1977)

$$g^{\mu\nu}\frac{\partial S}{\partial X^\mu}\frac{\partial S}{\partial X^\nu}+\mu^2=0 \tag{3}$$

we have

$$\begin{aligned}a^2\sin^2\theta\left(\frac{\partial S}{\partial v}\right)^2&+\Delta\left(\frac{\partial S}{\partial r}\right)^2+\left(\frac{\partial S}{\partial \theta}\right)^2\\&+\frac{1}{\sin^2\theta}\left(\frac{\partial S}{\partial \phi}\right)^2+2(r^2+a^2)\frac{\partial S}{\partial v}\frac{\partial S}{\partial r}\\&+2a\frac{\partial S}{\partial v}\frac{\partial S}{\partial \phi}+2a\frac{\partial S}{\partial r}\frac{\partial S}{\partial \phi}+\mu^2(r^2+a^2\cos^2\theta)=0\end{aligned} \tag{4}$$

where $S = S(v, r, \theta, \phi)$ is the master function, and μ is the mass of the particle.

Given the generalized tortoise coordinate transformation as (Zhao and Dai, 1991)

$$\begin{aligned}r_* &= r+\frac{1}{2k}\ln[r-r_H(v,\theta)]\\v_* &= v-v_0\\\theta_* &= \theta-\theta_0\end{aligned} \tag{5}$$

we have

$$\begin{aligned}\frac{\partial}{\partial r}&=\frac{2k(r-r_H)+1}{2k(r-r_H)}\frac{\partial}{\partial r_*}\\\frac{\partial}{\partial v}&=\frac{\partial}{\partial v_*}-\frac{\dot r_H}{2k(r-r_H)}\frac{\partial}{\partial r_*}\\\frac{\partial}{\partial \theta}&=\frac{\partial}{\partial \theta_*}-\frac{r'_H}{2k(r-r_H)}\frac{\partial}{\partial r_*}\end{aligned} \tag{6}$$

where $\dot r_H = \partial r_H/\partial v$, $r'_H = \partial r_H/\partial\theta$. Then equation (4) can be written as

$$\begin{aligned}a^2\sin^2\theta&\left[\frac{\partial S}{\partial \theta_*}-\frac{\dot r_H}{2k(r-r_H)}\frac{\partial S}{\partial r_*}\right]^2-\Delta\left[\frac{2k(r-r_H)+1}{2k(r-r_H)}\frac{\partial S}{\partial r_*}\right]^2\\&+\left[\frac{\partial S}{\partial \theta_*}-\frac{r'_H}{2k(r-r_H)}\frac{\partial S}{\partial r_*}\right]^2\end{aligned}$$

$$+ 2(r^2 + a^2)\left[\frac{\partial S}{\partial v_*} - \frac{\dot r_H}{2k(r - r_H)}\frac{\partial S}{\partial r_*}\right]\left[\frac{2k(r - r_H) + 1}{2k(r - r_H)}\frac{\partial S}{\partial r_*}\right]$$

$$+ \frac{1}{\sin^2\theta}\left(\frac{\partial S}{\partial \phi}\right)^2 + 2a\frac{\partial S}{\partial \phi}\left[\frac{\partial S}{\partial v_*} - \frac{\dot r_H}{2k(r - r_H)}\frac{\partial S}{\partial r_*}\right]$$

$$+ 2a\frac{\partial S}{\partial \phi}\left[\frac{2k(r - r_H) + 1}{2k(r - r_H)}\frac{\partial S}{\partial r_*}\right] + \mu^2(r^2 + a^2\cos^2\theta) = 0 \quad (7)$$

There exists a Killing vector $(\partial/\partial\phi)^\alpha$ in the space-time, so that

$$p_3 = m = \partial S/\partial\phi = \text{const} \quad (8)$$

Let us define

$$\omega = -\frac{\partial S}{\partial v_*}, \qquad p_2 = \frac{\partial S}{\partial \theta_*} \quad (9)$$

Equation (7) can be reduced to

$$\left(\frac{\partial S}{\partial r_*}\right)^2 \{a^2\sin^2\theta\, \dot r_H^2 + \Delta[2k(r - r_H) + 1]^2$$

$$+ (r_H')^2 - 2(r^2 + a^2)\dot r_H[2k(r - r_H) + 1]\}$$

$$+ \left(\frac{\partial S}{\partial r_*}\right)(2a\sin^2\theta\, \dot r_H\omega - 2\omega(r^2 + a^2)[2k(r - r_H) + 1]$$

$$- \{2am\dot r_H + 2p_2 r_H' - 2am[2k(r - r_H) + 1]\})2k(r - r_H)$$

$$+ [2k(r - r_H)]^2\left[p_2^2 + a^2\omega^2\sin^2\theta\right.$$

$$\left. + \frac{m^2}{\sin^2\theta} - 2am\omega + \mu^2(r^2 + a^2\cos^2\theta)\right] = 0 \quad (10)$$

Its solutions are

$$\frac{\partial S}{\partial r_*} = \frac{2k(r - r_H)B \pm 2k(r - r_H)(B^2 - AC)^{1/2}}{A} \quad (11)$$

where

$$A = a^2\dot r_H^2\sin^2\theta + \Delta[2k(r - r_H) + 1]^2 + (r_H')^2$$

$$- 2(r^2 + a^2)\dot r_H[2k(r - r_H) + 1]$$

$$B = -\omega a^2 \dot{r}_H \sin^2\theta + \omega(r^2 + a^2)[2k(r - r_H) + 1] + [am\dot{r}_H + p_2 r'_H - am[2k(r - r_H) + 1]]$$

$$C = p_2^2 + a^2\omega^2 \sin^2\theta + \frac{m^2}{\sin^2\theta} - 2am\omega + \mu^2(r^2 + a^2\cos^2\theta) \qquad (12)$$

Because both s and $\partial S/\partial r_*$ are real numbers, we get

$$B^2 - AC \geq 0 \qquad (13)$$

That is,

$$(-a^2\dot{r}_H\omega \sin^2\theta + \omega(r^2 + a^2)[2k(r - r_H) + 1] + \{am\dot{r}_H - am[2k(r - r_H) + 1] + p_2 r'_H\})^2 - A\left[a^2\omega^2 \sin^2\theta - 2am\omega + p_2^2 + \frac{m^2}{\sin^2\theta} + \mu^2(r^2 + a^2 \cos^2\theta)\right] \geq 0 \qquad (14)$$

This is the relation that the energy levels of Dirac particles have to satisfy in space-time. Let us adopt the equality in (14); it can be written as

$$\omega^{\pm} = \frac{-b}{\{[2k(r - r_H) + 1](r^2 + a^2) - \dot{r}_H a^2 \sin^2\theta\}^2 - Aa^2 \sin^2\theta} \pm E \qquad (15)$$

where

$$b = Aam + \{(r^2 + a^2)[2k(r - r_H) + 1] - \dot{r}_H a^2 \sin^2\theta\} \times \{am\dot{r}_H - am[2k(r - r_H) + 1] + p_2 r'_H\} \qquad (16)$$

$$E = [b^2 - (\{(r^2 + a^2)[2k(r - r_H) + 1] - a^2\dot{r}_H \sin^2\theta\}^2 - Aa^2 \sin^2\theta) \times (\{am\dot{r}_H - am[2k(r - r_H) + 1] + p_2 r'_H\}^2 - A(\mu^2 r^2 + K))]^{1/2} \times (\{(r^2 + a^2)[2k(r - r_H) + 1] - \dot{r}_H a^2 \sin^2\theta\}^2 - Aa^2 \sin^2\theta)^{-1} \qquad (17)$$

$$K = p_2^2 + \mu^2 a^2\cos^2\theta + \frac{m^2}{\sin^2\theta} \qquad (18)$$

The distribution of the energy levels of the Dirac vacuum is given by

$$\omega \geq \omega^+ \qquad (19)$$

and

$$\omega \leq \omega^- \tag{20}$$

The forbidden region is

$$\omega^- < \omega < \omega^+ \tag{21}$$

the width of which is

$$\omega^+ - \omega^- = 2E \tag{22}$$

Now let us consider the case near the event horizon r_H. We have

$$\lim_{r \to r_H} A = (r_H^2 + a^2)(1 - 2\dot{r}_H) + \dot{r}_H^2 a^2 \sin^2\theta - 2Mr_H + (r'_H)^2 = 0 \tag{23}$$

This is just the null surface condition, so the limit of A is zero. With (15)–(18) we get

$$\lim_{r \to r_H} 2E = 0 \tag{24}$$

$$\omega_0 = \lim_{r \to r_H} \omega^+ = \lim_{r \to r_H} \omega^- = \frac{am - am\dot{r}_H - p_2 r'_H}{(r_H^2 + a^2) - \dot{r}_H a^2 \sin^2\theta} \tag{25}$$

This means that the width of the forbidden region vanishes at the event horizon. When r goes to infinity we have

$$\omega^{\pm} \to \pm\mu \tag{26}$$

The distribution of the Dirac energy levels goes to that in the Minkowski space-time. Relations (15)–(21) show the distribution of the Dirac energy levels in the nonstationary Kerr space-time. There exists a crossing of the positive and negative energy levels near the event horizon. The Starobinsky–Unruh process (spontaneous radiation) must occur when $\omega_0 > +\mu$. This means that there is radiation from the region near the event horizon. This quantum effect is nonthermal. It is independent of temperature of the black hole. The maximum energy of a particle in this effect is ω_0. It is very interesting that ω_0 depends not only on the dragging velocity Ω_H [$\sim a/(r_H^2 + a^2)$], but also the evaporation rate ($\sim \dot{r}_H$) as well as the event horizon shape ($\sim r'_H$) of the black hole.

When $\dot{r}_H = 0$ and $r'_H = 0$, we have

$$\omega^{\pm} = \frac{am(r^2 + a^2 - \Delta)}{(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta} \pm \frac{\{\Delta a^2 m^2[a^2\sin^2\theta + \Delta - 2(r^2 + a^2)] + \Delta(\mu^2 r^2 + K)[(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta]\}^{1/2}}{(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta} \tag{27}$$

$$\omega_0 = \lim_{r \to r_H} \omega^{\pm} = \frac{am}{r_H^2 + a^2} \tag{28}$$

They reduce to the well-known stationary Kerr space-time results.

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